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Locating Resonances for Axiom *A* Dynamical Systems

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For a class of differentiable dynamical systems (called Axiom A systems) it has been shown by Pollicott and the author that correlation functions have Fourier transforms which are meromorphic in a strip. The poles (or resonances) are, however, not easy to locate. This note reviews the results which are known and discusses a simple model where the position of resonances can be estimated. The effect of noise is also discussed.

KEY WORDS: Correlation function; power spectrum; dynamical systems; axiom A; resonance; Gibbs state; Liapunov exponent; noise.

1. INTRODUCTION

A differentiable dynamical system (f^t) on a compact manifold M is a family of differentiable maps f^t: $M \to M$ such that $f^0 = \text{identity}$ and $f^{s+t} = f^s f^t$. We allow t to vary over the reals (continuous time case) or the integers (discrete time case), possibly with the restriction $t \geqslant 0$. For suitable x_0 the following time averages exist

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T A(f^t x_0)\, dt = \langle A \rangle = \int \rho(dx)\, A(x) \tag{1}$$

for all continuous A: $M \to \mathbb{R}$, and (1) defines a probability measure ρ, ergodic with respect to (f^t). In the discrete time case, the integral in (1) is replaced by a sum. We assume that a natural choice of ρ has been made, corresponding to the fact that time averages are often well defined in

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physical applications. If $B, C\colon M \to \mathbb{R}$ are differentiable functions, a *correlation function* ρ_{BC} is defined by

$$\begin{aligned}\rho_{BC}(t) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T B(f^{t+\tau}x_0)\, C(f^\tau x_0)\, d\tau - \langle B\rangle\langle C\rangle \\ &= \rho((B\circ f^t)\, C) - \rho(B)\, \rho(C).\end{aligned}$$

Its Fourier transform is

$$\hat{\rho}_{BC}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, \rho_{BC}(t)$$

If $B = C$, this Fourier transform is known as the *power spectrum* of the signal $t \to B(f^t x_0)$. Note that, in the discrete time case, $\hat{\rho}_{BC}$ becomes a periodic function of period 2π.

For a particular class of dynamical systems, called *Axiom A Systems*, it has been possible to prove that $\hat{\rho}_{BC}$ is meromorphic in a strip $|\mathrm{Im}\,\omega| < \delta$ (see Pollicott[12] and Ruelle[17,18]). The poles of $\hat{\rho}_{BC}$ may be interpreted as *resonances* of the dynamical system; their residues depend on B and C, but their positions do not. Unfortunately, little is known in general on the position of the resonances. For *mixing* systems (i.e., if $\rho_{BC} \to 0$ at infinity for all choices of B, C) there are no poles of $\hat{\rho}_{BC}$ on the real axis. In the discrete time mixing case, periodicity of $\hat{\rho}_{BC}$ implies that there are no poles in the strip $|\mathrm{Im}\,\omega| < \varepsilon$ for sufficiently small $\varepsilon > 0$ (this corresponds to the known fact that $\rho_{BC} \to 0$ exponentially at infinity, see Ruelle[13]). In the continuous time case, also called *flow*, the poles can come arbitrarily close to the real axis (i.e., for a pole ω, $\mathrm{Im}\,\omega$ may be arbitrarily small when $\mathrm{Re}\,\omega$ is sufficiently large). Unfortunately, the only small class of examples known at present in which the poles come close to the real axis does not correspond to an attractor. (See Ruelle[16]; naturally here ρ_{BC} does not tend exponentially to zero at infinity.) On the other hand, only one small class of examples is known where ρ_{BC} does tend to zero exponentially at infinity, and there are no poles of ρ_{BC} in a strip $|\mathrm{Im}\,\omega| < \varepsilon$. (This was proved by Collet, Epstein, and Gallavotti[4] for the geodesic flow on a manifold of *constant* negative curvature. Nothing is known when the curvature is nonconstant.)

From the above, it is clear that there is a serious lack of examples of Axiom A systems for which the resonances can be located. In the present note we shall discuss examples for which the position of resonances is accessible numerically and even analytically.

2. EXPANDING MAPS ASSOCIATED WITH AXIOM A SYSTEMS

The main reason why Axiom A dynamical systems are much more accessible to study than general differentiable dynamical systems is because of the existence of a technical tool called *symbolic dynamics* (based on *Markov partitions*; see Bowen[2]). Here we shall bypass the study of Axiom A systems and directly discuss a problem reformulated with the help of a Markov partition.

To be specific, we start from an Axiom A *flow* on M, and first restrict it to a *basic set* Λ (see Smale[19] and Bowen[1] for definitions). A differentiable function A: $M \to \mathbb{R}$ is also given, which specifies the ergodic measure ρ: ρ is the *Gibbs state* for $A|\Lambda$. One can show that the time averages (1) for Lebesgue almost all x_0 near an Axiom A attractor are given by a particular Gibbs state ρ called *SRB measure.*[2] On the other hand, $A=0$ yields the *measure of maximum entropy* on Λ. Here, however, we proceed with general Gibbs states. To the above setup, a Markov partition associates a quadruple (Ω, g, r, a) as follows: Ω is a compact metric space, g: $\Omega \to \Omega$ is an expanding map (see below, g is usually not invertible), and r, a: $\Omega \to \mathbb{R}$ are Lipschitz continuous functions such that r is strictly positive. (Lipschitz continuous means that $|r(\xi)-r(\eta)| \leqslant \text{const}\ \text{dist}(\xi, \eta)$, and similarly for a). Consider now the set

$$\{(\xi, u) \in \Omega \times \mathbb{R}: 0 \leqslant u \leqslant r(\xi)\}$$

and identify $(\xi, r(\xi))$ to $(g\xi, 0)$ to obtain a space Ω^*. For $t \geqslant 0$ let

$$g^{*t}(\xi, u) = (g^n\xi, u+t-r(\xi)-r(g\xi)- \cdots -r(g^{n-1}\xi))$$

where $n \geqslant 0$ is the largest integer such that

$$u+t \geqslant r(\xi)+r(g\xi)+ \cdots +(g^{n-1}\xi)$$

We have just defined a *semiflow* (g^{*t}) on Ω^* which is the new guise of the original Axiom A flow (f^t) on Λ. The function a serves to define a measure on Ω corresponding to the Gibbs state defined by A on Λ.

How are Ω, g, r obtained? Let us first say that a Markov partition for the flow (f^t) (restricted to Λ) consists of a finite number of pieces of hypersurfaces transversal to the flow. We call $(t_{\alpha\beta})$ the matrix with elements $t_{\alpha\beta}=1$ if the orbit (f^t) can successively cross α and β, zero otherwise. Let $(\cdots \xi_{-1}, \xi_0, \xi_1, \xi_2 \cdots)$be the sequence of hypersurfaces successively crossed by the orbit $(f^t x)$ so that the first crossing at positive time is that of ξ_1. Up to some controllable ambiguities, it is possible to represent x by

[2] See Ref. 5 for details and references.

the sequence $\boldsymbol{\xi} = (\cdots \xi_{-1}, \xi_0, \xi_1, \xi_2 \cdots)$ and the time u elapsed since crossing ξ_0 (i.e., $f^{-u}x \in \xi_0$). Note that $0 \leqslant u \leqslant \mathbf{r}(\boldsymbol{\xi})$ where the *ceiling function* $\mathbf{r}$ is the time between crossings of hypersurfaces of the Markov partition. One can arrange that $\mathbf{r}$ depends only on the half infinite sequence $\xi = (\xi_0, \xi_1,...)$. We write then $\mathbf{r}(\boldsymbol{\xi}) = r(\xi)$. We also define Ω to be the space of half infinite sequences ξ, with the condition $t_{\xi_n \xi_{n+1}} = 1$ for all $n \geqslant 0$, and with the metric

$$d(\xi, \eta) = \exp[-K \max\{k: \xi_k = \eta_k\}] \tag{2}$$

for some $K > 0$. Then the map

$$g: (\xi_0, \xi_1, \xi_2,...) \to (\xi_1, \xi_2,...)$$

is expanding in the sense that there are $C > 1$ and $\varepsilon > 0$ such that $\operatorname{dist}(g\xi, g\eta) \geqslant C \operatorname{dist}(\xi, \eta)$ when $\operatorname{dist}(\xi, \eta) < \varepsilon$. For suitable K in (2) the functions r and a are Lipschitz (the exact way in which a is defined will not be of concern to us).

An important variation of the construction sketched above is when everything can be made analytic, i.e., when Ω can somehow be embedded in a complex manifold so that g, r, a extend to holomorphic functions (see Ruelle,[14] Mayer,[10] and Fried[6]). While this situation is admittedly special, it is interesting and particularly suited to numerical investigation as we shall see in the example of Section 4.

3. LOCATING RESONANCES

As we have said, the position of the poles of $\hat{\rho}_{BC}$ does not depend on B, C (but the residues depend on B, C, and might accidentally vanish). These poles come in complex conjugate pairs[3] $\omega_\alpha, \bar{\omega}_\alpha$ where we may assume that

$$0 < \operatorname{Im} \omega_\alpha < \delta$$

if the Axiom A flow (f^t) is mixing. We shall now characterize the ω_α in different ways in terms of (Ω, g, r, a).

First, corresponding to ρ, there is an invariant probability measure ρ^* for the semiflow (g^{*t}) on Ω^*, and we can define correlation functions

$$\rho^*_{UV}(t) = \rho^*((U \circ g^{*t}) \cdot V) - \rho^*(U)\, \rho^*(V)$$

[3] In fact, with ω_α, also $\bar{\omega}_\alpha$, $-\omega_\alpha$, $-\bar{\omega}_\alpha$ are poles of $\hat{\rho}_{BC}$.

When U, V are nice functions on Ω^* (say Lipschitz continuous), the Fourier transform

$$\hat{\rho}^*_{UV}(\omega) = \int_0^\infty dt\, e^{i\omega t} \rho^*_{UV}(t)$$

is holomorphic for $\operatorname{Im} \omega > 0$, meromorphic for $\operatorname{Im} \omega > -\delta$, and its poles are located at the same positions $\bar{\omega}_\alpha$ as the poles of $\hat{\rho}_{BC}$ with negative imaginary part. This first characterization of the ω_α, however, is not extremely useful.

A second characterization is in terms of an operator $\mathscr{L}$ (*transfer matrix*) acting on the Banach space of Lipschitz functions on Ω. We define

$$(\mathscr{L}_b \varphi)(\xi_1, \xi_2, ...) = \sum_{\xi_0} \varphi(\xi_0, \xi_1, \xi_2, ...) \exp b(\xi_0, \xi_1, \xi_2, ...) \tag{3}$$

where b is a complex Lipschitz function on Ω, and the sum is over those ξ_0 such that $t_{\xi_0 \xi_1} = 1$. [We assume that the matrix $(t_{\alpha\beta})$ to some power N has all its matrix elements >0 (mixing property). The Markov partition can always be chosen such that this is true.] One can show that there is a real number $P(\operatorname{Re} b)$ such that

(i) The part of the spectrum of $\mathscr{L}_b$ in $\{z: |z| > e^{P(\operatorname{Re} b) - K}\}$ (with K as in (2)) consists of discrete eigenvalues with finite multiplicities.

(ii) If b is real, then $e^{P(b)}$ is a simple eigenvalue, and there are no other eigenvalues with modulus $> e^{P(b)}$.

See Pollicot[11] for (i) and Ruelle[13] for (ii). See also Keller[7] for related results.

Let now $b = a - sr$, where a, r are as above, and s is a complex number. We shall be interested in those values of s such that $\mathscr{L}_{a-sr}$ has 1 as an eigenvalue. In particular P^* is such a value if P^* is real and $P(a - P^* r) = 0$. It is known that P^* exists and is unique; in the special case corresponding to SRB measures we have $P^* = 0$.

Proposition. Consider a mixing Axiom A flow (f^t). The poles ω_α of $\hat{\rho}_{BC}$ with $0 < \operatorname{Im} \omega_\alpha < \delta$ are precisely the numbers $i(P^* - s)$ where $s \neq P^*$ is such that $\operatorname{Re} s > P^* - \delta$ and $\mathscr{L}_{a-sr}$ has 1 as an eigenvalue.

Our third and last characterization of the ω_α will be in terms of zeta functions.

Let $l(\gamma)$ be the period of a periodic orbit γ of (f^t) contained in the basic set γ. We define

$$\zeta_A(s) = \prod_\gamma \left[1 - \exp \int_0^{l(\gamma)} (A(f^t x_\gamma) - s)\, dt \right]^{-1}$$

where x_γ is any point of γ; the product is over all periodic orbits γ and converges for sufficiently large Re s. One can similarly define a zeta function ζ for the semiflow (g^{*t}), and this can be rewritten in terms of the periodic points of g: $\Omega \to \Omega$. We have

$$\zeta(s) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\xi:\, g^n\xi = \xi} \exp \sum_{k=0}^{n-1} [a(g^k\xi) - sr(g^k\xi)]$$

which again converges for sufficiently large Re s. For suitably chosen δ one can prove the following: ζ_A and ζ extend meromorphically to $\operatorname{Re} s > P^* - \delta$ and have the same poles in this region; for $\operatorname{Re} s > P^*$ both ζ_A and ζ are holomorphic.

Proposition. Consider a mixing Axiom A flow (f^t). The poles ω_α of $\hat{\rho}_{BC}$ with $0 < \operatorname{Im} \omega_\alpha < \delta$ are precisely the numbers $i(P^* - s)$ where $s \neq P^*$ is such that $\operatorname{Re} s > P^* - \delta$ and s is a pole of ζ_A or ζ.

The reader must be referred to Pollicott[12] and Ruelle[18] for the proofs, which are not very easy.

Note that if $0 < \varepsilon < \delta$, $\hat{\rho}_{BC}$ is analytic in the strip $|\operatorname{Im} \omega| < \varepsilon$ if and only if the numbers $s \neq P^*$ such that $\mathscr{L}_{a-sr}$ has eigenvalue 1 (or at which $\zeta(s)$ has a pole) satisfy $\operatorname{Re} s \leqslant P^* - \varepsilon$.

4. AN EXAMPLE: EXPANDING MAPS OF THE CIRCLE

We discuss here a simple situation, where the transfer matrix $\mathscr{L}$ and the zeta function ζ can be written fairly explicitly, but the ω_α are still not easily located.

We take Ω to be a circle (i.e., the interval $[0, 2\pi]$ with 0 and 2π identified), g is multiplication by an integer $q \geqslant 2$ (i.e., $g\xi = q\xi \pmod{2\pi}$), a vanishes, and r is a real analytic periodic function. It would be easy to reformulate these data in terms of symbolic dynamics.[4] We prefer to use Ω, g, r as given, preserving the fact that g and r real analytic. Suppose that r is analytic in the strip $|\operatorname{Im} \xi| < \kappa$, and note that the operator $\mathscr{L}$ of (3) is now given by

$$\begin{aligned}(\mathscr{L}\varphi)(\xi) &= \sum_{\eta:\, g\eta = \xi} \varphi(\eta) \exp(-sr(\eta)) \\ &= \sum_{k=0}^{q-1} \varphi\left(\frac{\xi + 2k\pi}{q}\right) \exp\left(-sr\left(\frac{\xi + 2k\pi}{q}\right)\right) \end{aligned} \tag{4}$$

[4] Using the "Markov partition" of Ω into the intervals $[0, 2\pi/q]$,..., $[(q-1)\, 2\pi/q, 2\pi]$ we replace ξ by the sequence $(\xi_1, \xi_2,...)$ such that $\xi_n = p$ when $f^n\xi \in [p2\pi/q, (p+1)\, 2\pi/q]$. Note that $\xi_1\xi_2 \cdots$ is the representation of $\xi/2\pi$ in basis q.

If φ is periodic and analytic in the strip $|\mathrm{Im}\ \xi| < \lambda$, then $\mathscr{L}\varphi$ is periodic and analytic in the strip $|\mathrm{Im}\ \xi| < \min(\kappa, q\lambda)$. Choosing $\lambda < \kappa$, we see that $\mathscr{L}$ is analyticity improving. One can show easily that $\mathscr{L}$ is a compact operator on the Banach space (with the uniform norm) of periodic functions which are continuous for $|\mathrm{Im}\ \xi| \leqslant \lambda$, and holomorphic for $|\mathrm{Im}\ \xi| < \lambda$. [For instance, if $r = \log(1 + \alpha^2 - 2\alpha \cos \xi)$, with $\alpha > 0$, we may take $\kappa = |\log \alpha|$.] Note that, writing

$$e_k(\xi) = e^{ik\xi}, \qquad e^{-sr} = \sum_{k=-\infty}^{\infty} a_k e_k$$

we have

$$\mathscr{L} e_m = q \sum_{k=-\infty}^{\infty} a_{kq-m} e_k$$

For the zeta function we have

$$\begin{aligned}
\zeta(s) &= \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\xi:\, g^n\xi = \xi} \exp \sum_{k=0}^{n-1} -sr(g^k\xi) \\
&= \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{q^n-1} \exp\left[-s \sum_{k=0}^{n-1} r\left(m \frac{2\pi q^k}{q^n - 1} \right) \right] \\
&= \prod_{n=1}^{\infty} \prod_{\gamma} \left[1 - \exp\left[-s \sum_{k=1}^{n-1} r(q^k \xi_\gamma) \right] \right]^{-1}
\end{aligned}$$

In the right-hand side, the second product is over all periodic orbits of period n for $\xi \to q\xi$ (mod 2π), and ξ_γ is an arbitrary element of γ [we may thus write $\xi_\gamma = (m/q^n - 1)\, 2\pi$, where $0 < m < q^n - 1$ and m does not divide $q^n - 1$]. The zeta function may be rewritten as

$$\zeta(s) = \frac{d_0(s)}{d_1(s)} \tag{5}$$

where

$$d_0(s) = \exp\left[-\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\xi:\, g^n\xi = \xi} \right] \frac{1}{\det(Dg^n - 1)} \exp \sum_{k=0}^{n-1} -sr(g^k\xi) \tag{6}$$

$$d_1(s) = \exp\left[-\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\xi:\, g^n\xi = \xi} \right] \frac{Dg^n}{\det(Dg^n - 1)} \exp \sum_{k=0}^{n-1} -sr(g^k\xi) \tag{7}$$

and Dg^n is the derivative of g^n at ξ, so that $Dg^n = q^n$ and $\det(Dg^n - 1) = q^n - 1$. Using an extension of the Fredholm theory due to Grothendieck we may interpret d_0 as the Fredholm determinant of $(1-\mathscr{M}_0)$, where the operator $\mathscr{M}_0$ acting on holomorphic functions is given by

$$(\mathscr{M}_0\varphi_0) = \sum_{\eta:\, g\eta=\xi} \varphi_0(\eta)\exp(-sr(\eta))$$

Similarly, d_1 is the Fredholm determinant of $(1-\mathscr{M}_1)$, where $\mathscr{M}_1$ is like $\mathscr{M}_0$, but acting on 1-forms:

$$(\mathscr{M}_1\varphi_1)(\xi) = \sum_{\eta:\, g\eta=\xi} D_\eta g\cdot\varphi_1(\eta)\exp(-sr(\eta))$$

The functions d_0 and d_1 are both entire analytic (see Ruelle[14] and Fried[6],[5] and the poles of ζ are given by the zeros of d_1.

Notice that when $r=1$, we have

$$d_1(s) = 1 - qe^{-s}$$

which vanishes at $\log q + m\cdot 2\pi i$, m integer. If r is close to 1 there will be zeros of d_1 at s_m close to $\log q + m\cdot 2\pi i$, with $s_0 = P^*$ and $\operatorname{Im} s_m < P^*$ for $k\neq 0$ if the system is mixing.

If we take $r = 1+\alpha\cos\xi$, a formal calculation shows that, to second order in α,

$$s_m = \log q + \frac{\alpha^2}{4}(\log q)^2 - \alpha^2\pi^2 m^2 + m\cdot\left(1+\frac{\alpha^2}{2}\log q\right)\cdot 2\pi i$$

This calculation is done by analogy with Ruelle[15] and Widom, Bensimon, Kadanoff, and Schenker.[20] If higher orders could be neglected (there is a problem of uniformity in m!), we see that in the present situation the Fourier transform $\hat\rho_{BC}$ would have no pole in the region $|\operatorname{Im}\omega| < \varepsilon$, with $\varepsilon\approx\alpha^2\pi^2$. [This would correspond to correlations decaying with time like $\exp(-\alpha^2\pi^2|t|)$.] One can hope in this case, and for more general r, to check numerically whether or not there is $\varepsilon>0$ such that $\operatorname{Im} s_m \leqslant P^* - \varepsilon$ for $m\neq 0$. [This check can be attempted either by looking at the eigenvalues of $\mathscr{L}$ or at the zeros of d_1].

Instead of the map $\xi\to\xi \pmod{2\pi}$ we may look at more general expanding maps g of the circle. We assume that g is real analytic and that the derivative $D_\xi g$ is everywhere >1. Much of what we have said above stays true; in particular, formulas (5), (6), (7) remain correct. It is

[5] Fried's paper (Ref. 6) contains a correction of the estimate of the order of the entire functions given in Ref. 14.

interesting to notice that if we choose $r = \log Dg$ we have the functional relation

$$d_1(s) = d_0(s-1)$$

so that the zeros and poles of ζ are related by a simple translation. In this case one can also prove that d_1 vanishes at $P^* = 1$. (This corresponds to the fact that the Hausdorff dimension of the circle Ω is 1, as shown in Ref. 15. Actually, the problems discussed here also arise for the zeta functions associated with Julia sets; see Refs. 15 and 20.)

5. CORRELATION FUNCTIONS IN THE PRESENCE OF NOISE

In this section we return to the definition of correlation functions. For large T,

$$\begin{aligned} \rho_{BC}(t) &\approx \frac{1}{T}\int_0^T B(f^{t+\tau}x_0)\, C(f^\tau x_0)\, d\tau - \langle B\rangle\langle C\rangle \\ &= \rho((B\circ f^t)\cdot C) - \rho(B)\,\rho(C) \end{aligned} \tag{8}$$

Both in the case of a physical experiment and of a computer simulation, there is some imprecision in $f^\tau x_0$ (noise for physical systems, roundoff errors—again treated as noise—for computer simulations). It is believed that this imprecision is what determines the choice of a *physical* ρ among many possible (f^t) ergodic probability measures. In other words, ρ is the zero noise limit of a stationary measure for a stochastic process obtained by adding noise to the deterministic time evolution (f^t). This can be made rigorous for Axiom A attractors (ρ is then the SRB measure, see Kiefer[8] and Young[21]) but is thought to be true with greater generality. (There are exceptions; see Ref. 5 for a more extended discussion.) In particular, ρ should satisfy the Pesin identity

$$\mathscr{L}(\rho) = \sum \text{positive characteristic exponents} \tag{9}$$

The *entropy* $h(\rho)$ and the *characteristic* (or Liapunov) *exponents* $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$ are defined in Ref. 5. Each λ_i is a possible value of the exponential rate of growth of $(f^t x_0)$ with time for a small perturbation δx_0 of the initial condition. For almost all x_0, the largest characteristic exponent λ_1 is observed. In general one has $h(\rho) \leqslant \Sigma$ positive λ_i. The meaning of the equality (9) has been discussed by Ledrappier and Young[9] (in the Axiom A case, it corresponds to $P^* = 0$).

For a chaotic system, the largest characteristic exponent λ_1 is >0 (we take this as definition of a chaotic system). If the level of noise is ν (and the size of the attractor is of order 1) the noise will swamp the time evolution $\tau \to f^\tau x$ after a time of the order of

$$|\log \nu|/\lambda_1$$

In view of what we have said the time T in (8) should satisfy

$$T \gg |\log \nu|/\lambda_1 \tag{10}$$

for the selection of ρ to be ensured. Condition (10) is usually easy to fulfill. A more difficult condition is that t in (8) should satisfy

$$t \ll |\log \nu|/\lambda_1 \tag{11}$$

Unless (11) holds, the computed correlation function will probably be dominated by the propagation of errors.

In fact, a heuristic argument shows that the propagation of errors, if (11) is not satisfied, produces an exponential decay of ρ_{BC}, with rate $-h(\rho)$. Indeed, the effect of noise at time t corresponds to averaging on unstable manifolds over a region of volume

$$\approx \nu \exp(t\ \Sigma \text{ positive } \lambda_i)$$

Assuming that the unstable manifold is evenly spread over the attractor, this yields an averaged $B - \langle B \rangle$ which tends to zero like $[\exp(t\ \Sigma \text{ positive } \lambda_i)]^{-1}$, i.e., like $\exp -th(\rho)$ if Pesin's identity (9) holds.

In conclusion, when (11) is not respected, the true decay of $\rho_{BC}(t)$ is combined with the decay due to the propagation of errors to give a decay at least as fast as $\exp -th(\rho)$. Conversely, to estimate the true decay of $\rho_{BC}(t)$ we have to respect (11); in the case of computer studies this may necessitate multiprecision calculations.

All this applies, for instance, to the discrete time dynamical system defined by the map

$$x \to ax(1-x)$$

of the interval $[0, 1]$ to itself, for $0 \leqslant a \leqslant 4$. In some cases for which $\lambda_1 > 0$, it is known that the correlations decay exponentially [for instance, if $a = 4$; see Collet[3] for more general results]. But Axiom A does not hold here, and it is not known if correlations decay exponentially in general.

Similarly, it would be interesting to have numerical evidence about the rate of decay of correlations for the Hénon attractor and the Lorentz attractor. Information about complex singularities (poles) of $\hat{\rho}_{BC}$ would be valuable, but probably difficult to obtain numerically.

Finally, let us return to examples of Axiom A flows. Let (f^t) be the geodesic flow on a compact surface S of constant curvature. If v is a smooth positive function on S, a perturbed flow $(\tilde{f}^t)$ is obtained by reparametrizing orbits so that the local velocity is v. The correlation functions for the Axiom A flow (f^t) are accessible to numerical investigations.

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